Comprehensive proof of the Greenberger-Horne-Zeilinger Theorem for the four-qubit system

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Greenberger-Horne-Zeilinger (GHZ) theorem asserts that there is a set of mutually commuting nonlocal observables with a common eigenstate on which those observables assume values that refute the attempt to assign values only required to have them by the local realism of Einstein, Podolsky, and Rosen (EPR). It is known that for a three-qubit system there is only one form of the GHZ-Mermin-like argument with equivalence up to a local unitary transformation, which is exactly Mermin's version of the GHZ theorem. In this paper, however, for a four-qubit system which was originally studied by GHZ, we show that there are nine distinct forms of the GHZ-Mermin-like argument. The proof is obtained by using some geometric invariants to characterize the sets of mutually commuting nonlocal spin observables on the four-qubit system. It is proved that there are at most nine elements (except for a different sign) in a set of mutually commuting nonlocal spin observables in the four-qubit system, and each GHZ-Mermin-like argument involves a set of at least five mutually commuting four-qubit nonlocal spin observables with a GHZ state as a common eigenstate in GHZ's theorem. Therefore, we present a complete construction of the GHZ theorem for the four-qubit system.

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I. INTRODUCTION

Bell's inequality [1] indicates that certain statistical correlations predicted by quantum mechanics for measurements on two-qubit ensembles cannot be understood within a realistic picture based on Einstein, Podolsky, and Rosen's (EPR's) notion of local realism [2]. There is an unsatisfactory feature in the derivation of Bell's inequality that such a local realistic and, consequently, classical picture can explain perfect correlations and is only in conflict with statistical prediction of quantum mechanics. Strikingly enough, the Greenberger-Horne-Zeilinger (GHZ's) theorem exhibits that the contradiction between quantum mechanics and local realistic theories arises even for definite predictions on a four-qubit system [3]. Mermin [4] subsequently refined the original GHZ argument on a three-qubit system. Let us recall that their approaches were characterized by the following premises:

(a) a set of mutually commuting nonlocal observables,

(b) a common eigenstate on which those observables assume values that refute the attempt to assign values only required to have them by EPR's local realism.

Based on this criterion of a GHZ-Mermin-like argument, we define a GHZ-Mermin experiment by a set of mutually commuting nonlocal observables with at least two different observables at each site. (Note that, a com-

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mon local observable does not provide a random selection of measurements and so plays no role in the GHZ-Mermin-type proof.) A GHZ-Mermin experiment presenting a GHZ-Mermin-like argument on a certain common eigenstate is said to be nontrivial. There is no nontrivial GHZ-Mermin experiment in the two-qubit system, while in the three-qubit system there is only one nontrivial GHZ-Mermin experiment (with equivalence up to a local unitary transformation), which is exactly Mermin's version of the GHZ theorem [5]. On the other hand, the GHZ-Mermin-like argument has been extended to n qubits [6], and to multiparty multilevel systems [7]. So far, however, no complete construction of nontrivial GHZ-Mermin experiments is presented beyond the three-qubit system as noted in [5], there are only partial results [8].

In this paper, we will construct all nontrivial GHZ-Mermin experiments of the four-qubit system, for which the GHZ-like argument was developed originally by GHZ [3]. It is proved that there are nine distinct forms of the GHZ-Mermin-like argument on the four-qubit system, and each GHZ-Mermin-like argument involves a set of at least five mutually commuting four-qubit nonlocal spin observables with a GHZ state as a common eigenstate in GHZ's theorem. Precisely, we obtain the following results.

(i) All four-qubit GHZ-Mermin experiments of at most four elements are trivial.

(ii) Four-qubit GHZ-Mermin experiments of five (6, 7, or 8) elements possess 11 (9, 5, or 3) different forms, two of which are nontrivial.

(iii) A four-qubit GHZ-Mermin experiment contains at most nine elements and, the experiments of nine elements have two different forms, one of which is trivial, while another one is nontrivial.

(iv) In every nontrivial GHZ-Mermin experiment for the four-qubit system, the associated states exhibiting an "all versus nothing" contradiction between quantum mechanics and EPR's local realism must be GHZ states.

Our proof is based on some subtle mathematical arguments. We first classify the equivalence of GHZ-Mermin experiments by two basic symmetries acting on them. Then, we define two geometric invariants for a GHZ-Mermin experiment, which can be used to distinguish two inequivalent experiments. These arguments can be easily extended to n qubits.

The structure of this paper is as follows. In Sec.II, we first prove a lemma on the structure of two commuting nonlocal spin observables of n qubits. Then, we discuss two basic symmetries ((S_1) and (S_2)) acting on GHZ-Mermin experiments of n qubits. By these two basic symmetries we define the equivalence of GHZ-Mermin experiments. We illustrate that a four-qubit GHZ-Mermin experiment of three elements must equivalently be one of three different forms. Finally, we define two geometric invariants (C-invariants and R-invariants) for a GHZ-Mermin experiment. These two geometric invariants are invariant under (S_1), (S_2), and local unitary transformations (LU). They paly a crucial role in the equivalence of GHZ-Mermin experiments. In Sec.III, we show that a four-qubit GHZ-Mermin experiment of four elements must equivalently be one of seven different forms. Then we prove that every four-qubit GHZ-Mermin experiment of three or four elements is trivial. In Sec.IV, we show that four-qubit GHZ-Mermin experiments of five (6, 7, or 8) elements possess 11 (9, 5, or 3) different forms. It is proved that in each case there are two nontrivial GHZ-Mermin experiments and, the associated states exhibiting an "all versus nothing" contradiction between quantum mechanics and EPR's local realism are GHZ states. In Sec.V, we prove that a four-qubit GHZ-Mermin experiment contains at most nine elements and the experiments of nine elements have two different forms, one of which is trivial while another one is nontrivial. Finally, in Sec.VI we give some concluding remarks and questions for further consideration.

II. GHZ-MERMIN EXPERIMENTS AND SYMMETRIES

Let us consider a system of n qubits labelled by $1, 2, \cdots, n$. Let A_j, A'_j denote spin observables on the jth qubit, $j = 1, 2, \cdots, n$. For $A_j^{(\prime)} = \vec{a}_j^{(\prime)} \cdot \vec{\sigma}_j$ $(1 \le j \le n)$, we write

$$(A_j, A'_j) = (\vec{a}_j, \vec{a}'_j), A_j \times A'_j = (\vec{a}_j \times \vec{a}'_j) \cdot \vec{\sigma}_j.$$

Here $\vec{\sigma}_j = (\sigma_x^j, \sigma_y^j, \sigma_z^j)$ are the Pauli matrices for the jth qubit; the vectors $\vec{a}_j^{(\prime)}$ are all unit vectors in $\mathbb{R}^3$. It is easy to check that

$$A_j A'_j = (A_j, A'_j) + i A_j \times A'_j, \tag{2.1}$$

$$A'_j A_j = (A_j, A'_j) - i A_j \times A'_j, \tag{2.2}$$

$$\|A_j \times A'_j\|^2 = 1 - (A_j, A'_j)^2. \tag{2.3}$$

Also, $A_j \times A'_j = 0$ if and only if $A_j = \pm A'_j$, i.e., A_j is parallel to A'_j; $(A_j, A'_j) = 0$ if and only if A_j is orthogonal to A'_j, denoted by $A_j \perp A'_j$.

We write $A_1^{(\prime)} \cdots A_n^{(\prime)}$, etc., as shorthand for $A_1^{(\prime)} \otimes \cdots \otimes A_n^{(\prime)}$. The following lemma clarifies the inner structure of mutually commuting nonlocal spin observables of the n-qubit system.

Lemma: Two nonlocal n-qubit spin observables $A_1 \cdots A_n$ and $A'_1 \cdots A'_n$ are commuting if and only if for every $j = 1, 2, \cdots, n$, A_j is either parallel or orthogonal to A'_j, and the number of sites at which the corresponding local spin observables are orthogonal to each other is even.

Proof. The sufficiency is clear. Indeed, by Eqs.(2.1) and (2.2), we have that $A_j A'_j = -A'_j A_j$ whenever

$(A_j, A'_j) = 0$. Since the number of elements of $\{j : A_j \perp A'_j\}$ is even, it is immediately concluded that $A_1 \cdots A_n$ and $A'_1 \cdots A'_n$ are commuting.

To prove the necessity, suppose that $A_1 \cdots A_n$ and $A'_1 \cdots A'_n$ are commuting. For every unit vector $|u_1\rangle \otimes \cdots \otimes |u_n\rangle$, one has

$$\prod_{j=1}^{n} \|A_j A'_j |u_j\rangle\|^2 = \prod_{j=1}^{n} \langle A_j A'_j u_j | A'_j A_j u_j\rangle.$$

By Eqs.(2.1) and (2.2), we have

$$\|A_j A'_j |u_j\rangle\|^2 = (A_j, A'_j)^2 + \|A_j \times A'_j |u_j\rangle\|^2,$$

$$\langle A_j A'_j u_j | A'_j A_j u_j\rangle = (A_j, A'_j)^2 - \|A_j \times A'_j |u_j\rangle\|^2$$
$$-2i(A_j, A'_j)\langle u_j | A_j \times A'_j |u_j\rangle.$$

Note that, if $A_j \times A'_j \neq 0$, there correspond to two eigenvalues $\pm\|A_j \times A'_j\|$ with the corresponding unit eigenvectors $|u_j^{\pm}\rangle$. In this case, we set $|u_j\rangle = (|u_j^+\rangle + |u_j^-\rangle)/\sqrt{2}$ and obtain $\langle u_j | A_j \times A'_j |u_j\rangle = 0$. Hence, we have

$$\prod_{j=1}^{n} \left[(A_j, A'_j)^2 + \|A_j \times A'_j |u_j\rangle\|^2\right]$$
$$= \prod_{j=1}^{n} \left[(A_j, A'_j)^2 - \|A_j \times A'_j |u_j\rangle\|^2\right].$$

This immediately concludes that either $A_j \times A'_j = 0$ or $(A_j, A'_j) = 0$ for each $j = 1, 2, \cdots, n$. On the other hand, by Eqs.(2.1) and (2.2) we have that $A_j A'_j = -A'_j A_j$ whenever $(A_j, A'_j) = 0$. Therefore, the number of elements of $A_j \perp A'_j$ is even.

The Lemma tells us that two commuting nonlocal spin observables of the n-qubit system have a nice structure, which has been used to clarify the geometric structure of GHZ-Mermin experiments of both two-qubit and three-qubit systems in [5]. For convenience, we reformulate the Lemma in the case of four qubits that two four-qubit nonlocal spin observables $A_1A_2A_3A_4$ and $A'_1A'_2A'_3A'_4$ are commuting if and only if one of the following conditions is satisfied:

(1) $A_1 = \pm A'_1, A_2 = \pm A'_2, A_3 = \pm A'_3, A_4 = \pm A'_4;$
(2) $A_1 = \pm A'_1, A_2 = \pm A'_2, (A_3, A'_3) = (A_4, A'_4) = 0;$
(3) $A_1 = \pm A'_1, A_3 = \pm A'_3, (A_2, A'_2) = (A_4, A'_4) = 0;$
(4) $A_1 = \pm A'_1, A_4 = \pm A'_4, (A_2, A'_2) = (A_3, A'_3) = 0;$
(5) $A_2 = \pm A'_2, A_3 = \pm A'_3, (A_1, A'_1) = (A_4, A'_4) = 0;$
(6) $A_2 = \pm A'_2, A_4 = \pm A'_4, (A_1, A'_1) = (A_3, A'_3) = 0;$
(7) $A_3 = \pm A'_3, A_4 = \pm A'_4, (A_1, A'_1) = (A_2, A'_2) = 0;$
(8) $(A_1, A'_1) = (A_2, A'_2) = (A_3, A'_3) = (A_4, A'_4) = 0.$

This concludes that

$$\{A_1A_2A_3A_4, A'_1A'_2A_3A_4, A_1A_2A'_3A'_4\}, \tag{2.4}$$

$$\{A_1A_2A_3A_4, A'_1A'_2A_3A_4, A''_1A''_2A'_3A'_4\}, \tag{2.5}$$

and

$$\{A_1A_2A_3A_4, A'_1A'_2A'_3A'_4, A''_1A''_2A''_3A''_4\}, \tag{2.6}$$

are all GHZ-Mermin experiments of the four-qubit system, where

$$(A_j, A'_j) = (A_j, A''_j) = (A'_j, A''_j) = 0$$

for $j = 1, 2, 3$. In this article, we need to clarify the geometric structure of GHZ-Mermin experiments of the four-qubit system.

Browsing through the sets of mutually commuting four-qubit nonlocal spin observables we quickly get the feeling that there are many rather similar ones, and also some sets which can be obtained in a rather trivial way (e.g., add a common element) from 2-qubit and 3-qubit ones. Hence, there are many equivalent GHZ-Mermin experiments. Here, we describe the grouping of GHZ-Mermin experiments into "essentially distinct ones." Some symmetries acting on GHZ-Mermin experiments are obvious. There are two basic symmetries leading to equivalent experiments as follows.

(S_1) Changing the labelling of the local observables at each site.

(S_2) Permuting systems.

Here, we define as equivalent two GHZ-Mermin experiments $\mathcal{A}$ and $\mathcal{B}$ if they can be transformed to each other by symmetrical actions (S_1) and (S_2) or local unitary operations (LU). In this case, we denote by $\mathcal{A} \cong \mathcal{B}$. For example,

$$\{A''_1A''_2A_3A_4, A'_1A'_2A_3A_4, A''_1A''_2A'_3A'_4\} \cong \text{Eq.}(2.4)$$

by changing the labelling of the local observables with $A_1 \longleftrightarrow A''_1$ and $A_2 \longleftrightarrow A''_2$. Also,

$$\{A_1A_2A_3A_4, A'_1A_2A_3A'_4, A''_1A'_2A'_3A''_4\} \cong \text{Eq.}(2.5)$$

by permuting the system with qubit 2 $\Longleftrightarrow$ qubit 4.

Since $\mathrm{SU}(2) \cong \mathrm{SO}(3)$ through $U^\dagger(\vec{a}\vec{\sigma})U = (R\vec{a})\vec{\sigma}$, there is a local unitary transformation U_j on the jth qubit such that $A_j = U_j^* \sigma_x^j U_j, A'_j = U_j^* \sigma_y^j U_j$, and $A''_j = U_j^* \sigma_z^j U_j$, provided $(A_j, A'_j) = (A_j, A''_j) = (A'_j, A''_j) = 0$. Then, the GHZ-Mermin experiments Eqs.(2.4)-(2.6) are respectively equivalent to

$$\{\sigma_x^1\sigma_x^2\sigma_x^3\sigma_x^4, \sigma_y^1\sigma_y^2\sigma_x^3\sigma_x^4, \sigma_x^1\sigma_x^2\sigma_y^3\sigma_y^4\}, \tag{2.7}$$

$$\{\sigma_x^1\sigma_x^2\sigma_x^3\sigma_x^4, \sigma_y^1\sigma_y^2\sigma_x^3\sigma_x^4, \sigma_z^1\sigma_z^2\sigma_y^3\sigma_y^4\}, \tag{2.8}$$

and

$$\{\sigma_x^1\sigma_x^2\sigma_x^3\sigma_x^4, \sigma_y^1\sigma_y^2\sigma_y^3\sigma_y^4, \sigma_z^1\sigma_z^2\sigma_z^3\sigma_z^4\}. \tag{2.9}$$

Moreover, since local observables are either parallel or orthogonal to each other in a GHZ-Mermin experiment of n qubits by the above Lemma, it then must be equivalent to a GHZ-Mermin experiment with each local observable taking one of σ_x, σ_y and σ_z. Therefore, for constructing a GHZ-Mermin experiment of n qubits we only need to choose σ_x, σ_y, or σ_z as local observables.

Clearly, Eqs.(2.7)-(2.9) possess different geometric structure. That is, there are two dichotomic observables per site in Eq.(2.7), two triads $(\sigma_x^1, \sigma_y^1, \sigma_z^1)$ and $(\sigma_x^2, \sigma_y^2, \sigma_z^2)$ in Eq.(2.8), and four ones in Eq.(2.9). Since symmetrical actions (S_1) and (S_2) and local unitary operations (LU) do not change the geometric structure of GHZ-Mermin experiments, Eqs.(2.7)-(2.9) are inequivalent to each other. Generally speaking, every GHZ-Mermin experiment has two geometric invariants. On one hand, the number of sites which has a triad is invariant under $(\mathrm{S}_1), (\mathrm{S}_2)$, and (LU), denoted by C. Clearly, $\mathrm{C} \leq n$ for the n-qubit system. On the another hand, for every element of the experiment there corresponds to the number of elements which are orthogonal to that element at two sites. The set of those numbers is also invariant under $(\mathrm{S}_1), (\mathrm{S}_2)$, and (LU), denoted by R. For example, the C and R invariants of Eq.(2.7) are respectively 0 and $(2,1,1)$, the ones of Eq.(2.8) are 2 and $(1,1,0)$, and the ones of Eq.(2.9) are 4 and $(0,0,0)$. Eqs.(2.7)-(2.9) have different geometric invariants. In the sequel, we show that each four-qubit GHZ-Mermin experiment of three elements must equivalently be one of the forms Eqs.(2.7)-(2.9) and hence, two four-qubit GHZ-Mermin experiments of three elements are equivalent if and only if they have the same geometric invariants.

To this end, by (S_1) we have that each GHZ-Mermin experiment of three elements must be one of the forms

$$\{\sigma_x^1\sigma_x^2\sigma_x^3\sigma_x^4, \sigma_y^1\sigma_y^2\sigma_x^3\sigma_x^4, \star\star\star\star\}, \tag{2.10}$$

$$\{\sigma_x^1\sigma_x^2\sigma_x^3\sigma_x^4, \sigma_y^1\sigma_y^2\sigma_y^3\sigma_y^4, \star\star\star\star\}, \tag{2.11}$$

because

$$\begin{gathered}
\{\sigma_x^1\sigma_x^2\sigma_x^3\sigma_x^4, \sigma_y^1\sigma_x^2\sigma_y^3\sigma_x^4, \star\star\star\star\}, \\
\{\sigma_x^1\sigma_x^2\sigma_x^3\sigma_x^4, \sigma_y^1\sigma_x^2\sigma_x^3\sigma_y^4, \star\star\star\star\}, \\
\{\sigma_x^1\sigma_x^2\sigma_x^3\sigma_x^4, \sigma_x^1\sigma_y^2\sigma_y^3\sigma_x^4, \star\star\star\star\}, \\
\{\sigma_x^1\sigma_x^2\sigma_x^3\sigma_x^4, \sigma_x^1\sigma_y^2\sigma_x^3\sigma_y^4, \star\star\star\star\}, \\
\{\sigma_x^1\sigma_x^2\sigma_x^3\sigma_x^4, \sigma_x^1\sigma_x^2\sigma_y^3\sigma_y^4, \star\star\star\star\},
\end{gathered}$$

are all equivalent to Eq.(2.10) by (S_2). Since there are at least two distinct observables at each site, Eq.(2.10) reduces to Eq.(2.7), Eq.(2.8), and

$$\{\sigma_x^1\sigma_x^2\sigma_x^3\sigma_x^4, \sigma_y^1\sigma_y^2\sigma_x^3\sigma_x^4, \sigma_y^1\sigma_y^2\sigma_y^3\sigma_y^4\}. \tag{2.12}$$

However, Eq.(2.12) is equivalent to Eq.(2.7) by (S_1) with $\sigma_x^1 \longleftrightarrow \sigma_y^1$ and $\sigma_x^2 \longleftrightarrow \sigma_y^2$.

On the other hand, Eq.(2.11) reduces to Eq.(2.9),

$$\{\sigma_x^1\sigma_x^2\sigma_x^3\sigma_x^4, \sigma_y^1\sigma_y^2\sigma_y^3\sigma_y^4, \sigma_z^1\sigma_z^2\sigma_y^3\sigma_y^4\} \cong \text{Eq.}(2.8),$$

and

$$\{\sigma_x^1\sigma_x^2\sigma_x^3\sigma_x^4, \sigma_y^1\sigma_y^2\sigma_y^3\sigma_y^4, \sigma_z^1\sigma_z^2\sigma_x^3\sigma_x^4\} \cong \text{Eq.}(2.8).$$

This concludes the required result.

III. TRIVIAL GHZ-MERMIN EXPERIMENTS

There are many trivial GHZ-Mermin experiments of four qubits. For example, GHZ-Mermin experiments Eqs.(2.7)-(2.9) are all trivial. Indeed, we will show in this section that each of Eqs.(2.7)-(2.9) is included in a GHZ-Mermin experiment of four elements and all four-qubit GHZ-Mermin experiments of four elements are trivial.

In the following, we write $\sigma_x^1\sigma_x^2\sigma_y^3\sigma_y^4$, etc., as shorthand for $xxyy$ or $x_1x_2y_3y_4$. We characterize all four-qubit GHZ-Mermin experiments of four elements as follows.

Proposition: A $\mathrm{GHZ-Mermin}$ *experiment of four elements for the four-qubit system must equivalently be one of the following forms:*

$$\{xxxx, yyxx, zzxx, xxyy\}, \tag{3.1}$$

$$\{xxxx, yyxx, yxyx, xxyy\}, \tag{3.2}$$

$$\{xxxx, yyxx, yxyx, yxxy\}, \tag{3.3}$$

$$\{xxxx, yyxx, yxyx, zzzy\}, \tag{3.4}$$

$$\{xxxx, yyxx, xxyy, yyyy\}, \tag{3.5}$$

$$\{xxxx, yyxx, xxyy, zzyy\}, \tag{3.6}$$

$$\{xxxx, yyxx, xxyy, zzzz\}, \tag{3.7}$$

$$\{xxxx, yyxx, zzyy, zzzz\}. \tag{3.8}$$

Moreover, the geometric invariants of Eqs.(3.1) – (3.8) *are illustrated in* Table I.

Proof: At first, a GHZ-Mermin experiment of four elements for the four-qubit system must equivalently be one of the forms

$$\{xxxx, yyxx, \star\star\star\star, \star\star\star\star\}, \tag{3.9}$$

$$\{xxxx, yyyy, \star\star\star\star, \star\star\star\star\}, \tag{3.10}$$

Then, Eq.(3.9) reduces to one of the following forms

$$\{xxxx, yyxx, zzxx, \star\star\star\star\}, \tag{3.9-1}$$

$$\{xxxx, yyxx, yxyx, \star\star\star\star\}, \tag{3.9-2}$$

$$\{xxxx, yyxx, xxyy, \star\star\star\star\}, \tag{3.9-3}$$

$$\{xxxx, yyxx, zzyy, \star\star\star\star\}, \tag{3.9-4}$$

because $\{xxxx, yyxx, yxxy, \star \quad \star \quad \star\star\}$, $\{xxxx, yyxx, xyyx, \star\star\star\star\}$, and $\{xxxx, yyxx, xyxy, \star\star\star\star\}$ are all equivalent to Eq.(3.9-2), as well as $\{xxxx, yyxx, yyyy, \star\star\star\star\} \cong \text{Eq.}(3.9-3)$.

(1) From Eq.(3.9-1) we obtain Eq.(3.1) and

$$\begin{aligned}\{xxxx, yyxx, zzxx, yyyy\} &\cong \text{Eq.}(3.1),\\ \{xxxx, yyxx, zzxx, zzyy\} &\cong \text{Eq.}(3.1).\end{aligned}$$

(2) From Eq.(3.9-2) we obtain Eqs.(3.2)-(3.4) and

$$\begin{aligned}\{xxxx, yyxx, yxyx, xyxy\} &\cong \text{Eq.}(3.2),\\ \{xxxx, yyxx, yxyx, yyyy\} &\cong \text{Eq.}(3.2).\end{aligned}$$

(3) From Eq.(3.9-3) we obtain that Eqs.(3.1), (3.2), (3.5)-(3.7), and

$$\begin{aligned}\{xxxx, yyxx, xxyy, xxzz\} &\cong \text{Eq.}(3.1),\\ \{xxxx, yyxx, xxyy, yxxy\} &\cong \text{Eq.}(3.2),\\ \{xxxx, yyxx, xxyy, xyyx\} &\cong \text{Eq.}(3.2),\\ \{xxxx, yyxx, xxyy, xyxy\} &\cong \text{Eq.}(3.2),\\ \{xxxx, yyxx, xxyy, yyzz\} &\cong \text{Eq.}(3.6).\end{aligned}$$

(4) From Eq.(3.9-4) we obtain that Eqs.(3.6), (3.8), and

$$\begin{aligned}\{xxxx, yyxx, zzyy, zzxx\} &\cong \text{Eq.}(3.1),\\ \{xxxx, yyxx, zzyy, yyyy\} &\cong \text{Eq.}(3.6),\\ \{xxxx, yyxx, zzyy, yyzz\} &\cong \text{Eq.}(3.7),\\ \{xxxx, yyxx, zzyy, xxzz\} &\cong \text{Eq.}(3.7).\end{aligned}$$

On the other hand, since

$$\begin{aligned}&\{xxxx, yyyy, yyxx, \star\star\star\star\},\\ &\{xxxx, yyyy, zzxx, \star\star\star\star\},\\ &\{xxxx, yyyy, zzyy, \star\star\star\star\},\end{aligned}$$

and their variants are all included in Eq.(3.9), it is concluded that Eq.(3.10) reduces to

$$\{xxxx, yyyy, zzzz, \star\star\star\star\}, \tag{3.10-1}$$

TABLE I: Here, we denote by $j(=1,2,3,4)$ the j-th element of the experiments. The numbers in the C column are C-invariants, while the numbers in $1-4$'s columns are R-invariants.

	C	1	2	3	4
(3.1)	2	3	2	2	1
(3.2)	0	3	2	3	2
(3.3)	0	3	3	3	3
(3.4)	3	2	2	2	0
(3.5)	0	2	2	2	2
(3.6)	2	2	1	2	1
(3.7)	4	2	1	1	0
(3.8)	4	1	1	1	1

.From Eq.(3.10-1) we obtain

$$\begin{aligned}\{xxxx, yyyy, zzzz, yyxx\} &\cong \text{Eq.}(3.7),\\ \{xxxx, yyyy, zzzz, zzxx\} &\cong \text{Eq.}(3.7),\\ \{xxxx, yyyy, zzzz, zzyy\} &\cong \text{Eq.}(3.7).\end{aligned}$$

The proof is complete.

From the above Proposition, it suffices to consider Eqs.(3.1)-(3.8) for showing that every GHZ-Mermin experiments of four elements for the four-qubit system is trivial. Let us recall that the scenario for the GHZ-Mermin proof is the following: Particles 1, 2, 3, and 4 move away from each other. At a given time, an observer, Alice, has access to particle 1, a second observer, Bob, has access to particle 2, a third observer, Charlie, has access to particle 3, and a fourth observer, Davis, has access to particle 4. For example, in the case of Eq.(3.5), by introducing $(\cdot)$ to separate operators that can be viewed as EPR's local elements of reality and for any common eigenstate $|\varphi\rangle$ of Eq.(3.5), we have

$$\begin{aligned}x_1 \cdot x_2 \cdot x_3 \cdot x_4|\varphi\rangle &= \varepsilon_1|\varphi\rangle,\\ y_1 \cdot y_2 \cdot x_3 \cdot x_4|\varphi\rangle &= \varepsilon_2|\varphi\rangle,\\ x_1 \cdot x_2 \cdot y_3 \cdot y_4|\varphi\rangle &= \varepsilon_3|\varphi\rangle,\\ y_1 \cdot y_2 \cdot y_3 \cdot y_4|\varphi\rangle &= \varepsilon_4|\varphi\rangle,\end{aligned}$$

where $\varepsilon_j = \pm 1$. According to EPR's criterion of local realism [2], Eq.(3.5) allows Alice, Bob, Charlie, and Davis to predict the following relations between the values of the elements of reality:

$$\begin{aligned}\nu(x_1)\nu(x_2)\nu(x_3)\nu(x_4) &= \varepsilon_1,\\ \nu(y_1)\nu(y_2)\nu(x_3)\nu(x_4) &= \varepsilon_2,\\ \nu(x_1)\nu(x_2)\nu(y_3)\nu(y_4) &= \varepsilon_3,\\ \nu(y_1)\nu(y_2)\nu(y_3)\nu(y_4) &= \varepsilon_4.\end{aligned}$$

Since $(x_1x_2x_3x_4) \times (y_1y_2x_3x_4) \times (x_1x_2y_3y_4) \times (y_1y_2y_3y_4) = 1$, we have that $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = 1$. In this case, one can assign values $\nu(x_1) = \varepsilon_1$, $\nu(y_1) = \varepsilon_2, \nu(y_3) = \varepsilon_1\varepsilon_3$, and the remaining ones $\nu(\cdot) = 1$. Thus, the GHZ-Mermin proof is nullified in the case of Eq.(3.5).

TABLE II: In every case, we can assign the values as in the case of Eq.(3.5), the elements not indicated all take the value $\nu(\cdot) = 1$.

(3.1)	$\nu(x_1) = \varepsilon_1$	$\nu(y_1) = \varepsilon_2$	$\nu(z_1) = \varepsilon_3$	$\nu(y_3) = \varepsilon_1\varepsilon_4$
(3.2)	$\nu(x_1) = \varepsilon_1$	$\nu(y_2) = \varepsilon_2\varepsilon_3$	$\nu(y_1) = \varepsilon_3$	$\nu(y_4) = \varepsilon_1\varepsilon_4$
(3.3)	$\nu(x_1) = \varepsilon_1$	$\nu(y_2) = \varepsilon_2$	$\nu(y_3) = \varepsilon_3$	$\nu(y_4) = \varepsilon_4$
(3.4)	$\nu(x_1) = \varepsilon_1$	$\nu(y_2) = \varepsilon_2$	$\nu(y_3) = \varepsilon_3$	$\nu(z_1) = \varepsilon_4$
(3.6)	$\nu(x_1) = \varepsilon_1$	$\nu(y_1) = \varepsilon_2$	$\nu(y_3) = \varepsilon_1\varepsilon_3$	$\nu(z_1) = \varepsilon_1\varepsilon_3\varepsilon_4$
(3.7)	$\nu(x_1) = \varepsilon_1$	$\nu(y_1) = \varepsilon_2$	$\nu(y_3) = \varepsilon_1\varepsilon_3$	$\nu(z_1) = \varepsilon_4$
(3.8)	$\nu(x_1) = \varepsilon_1$	$\nu(y_1) = \varepsilon_2$	$\nu(y_3) = \varepsilon_3$	$\nu(z_3) = \varepsilon_4$

The other cases are illustrated in Table II.

Finally, we note that Eq.(2.7) is included in Eq.(3.1), Eq.(2.8) in Eq.(3.6), and Eq.(2.9) (equivalently) in Eq.(3.7). This concludes that the four-qubit GHZ-Mermin experiments of three elements are all trivial. Therefore, a nontrivial GHZ-Mermin experiment of four qubits must have at least five elements.

IV. NONTRIVIAL GHZ-MERMIN EXPERIMENTS

In this section, we will present a complete construction of nontrivial four-qubit GHZ-Mermin experiments of five (6, 7, 8) elements. We show that the experiments of five (6, 7, 8) elements possess 11 (9, 5, 3) different forms. It is proved that in each case there are two nontrivial GHZ-Mermin experiments and, the associated states exhibiting an "all versus nothing" contradiction between quantum mechanics and EPR's local realism are GHZ states.

A. The case of five elements

We first characterize all four-qubit GHZ-Mermin experiments of five elements as follows.

Proposition: A $\mathrm{GHZ-Mermin}$ *experiment of five elements for the four-qubit system must equivalently be one of the following forms:*

$$\{xxxx, yyxx, zzxx, xxyy, xxzz\}, \tag{4A.1}$$

$$\{xxxx, yyxx, zzxx, xxyy, yyyy\}, \tag{4A.2}$$

$$\{xxxx, yyxx, zzxx, xxyy, yyzz\}, \tag{4A.3}$$

$$\{xxxx, yyxx, yxyx, xxyy, yxxy\}, \tag{4A.4}$$

$$\{xxxx, yyxx, yxyx, xxyy, xyxy\}, \tag{4A.5}$$

TABLE III: The numbers in the C line are C-invariants, while the numbers in $1-5$'s lines are R-invariants.

4A	1	2	3	4	5	6	7	8	9	10	11
C	4	2	4	0	0	4	4	3	4	4	4
1	4	3	3	4	4	3	3	3	2	2	2
2	2	3	3	3	3	2	3	3	2	2	2
3	2	2	2	4	3	3	3	3	2	2	1
4	2	2	1	3	3	2	3	0	2	1	1
5	2	2	1	4	3	0	0	3	0	1	2

$$\{xxxx, yyxx, yxyx, xxyy, zzzz\}, \tag{4A.6}$$

$$\{xxxx, yyxx, yxyx, yxxy, zzzz\}, \tag{4A.7}$$

$$\{xxxx, yyxx, yxyx, zzzy, xyyx\}, \tag{4A.8}$$

$$\{xxxx, yyxx, xxyy, yyyy, zzzz\}, \tag{4A.9}$$

$$\{xxxx, yyxx, xxyy, zzyy, yyzz\}, \tag{4A.10}$$

$$\{xxxx, yyxx, xxyy, zzzz, yyzz\}. \tag{4A.11}$$

Moreover, the geometric invariants of Eqs.(4A.1) $-$ (4A.11) *are illustrated in* Table III.

Proof: By repeating the proof of the Proposition in Section III, we find that every subset of four elements in a GHZ-Mermin experiment of five elements for the four-qubit system is a GHZ-Mermin experiment, i.e., a set of four mutually commuting nonlocal spin observables with at least two different observables at each site. By the Proposition in Sec.III, this concludes that a four-qubit GHZ-Mermin experiment of five elements must equivalently be one of the forms

$$\{xxxx, yyxx, zzxx, xxyy, \star\star\star\star\}, \tag{4A.12}$$

$$\{xxxx, yyxx, yxyx, xxyy, \star\star\star\star\}, \tag{4A.13}$$

$$\{xxxx, yyxx, yxyx, yxxy, \star\star\star\star\}, \tag{4A.14}$$

$$\{xxxx, yyxx, yxyx, zzzy, \star\star\star\star\}, \tag{4A.15}$$

$$\{xxxx, yyxx, xxyy, yyyy, \star\star\star\star\}, \tag{4A.16}$$

$$\{xxxx, yyxx, xxyy, zzyy, \star\star\star\star\}, \tag{4A.17}$$

$$\{xxxx, yyxx, xxyy, zzzz, \star\star\star\star\}, \tag{4A.18}$$

$$\{xxxx, yyxx, zzyy, zzzz, \star\star\star\star\}. \tag{4A.19}$$

(1) From Eq.(4A.12), we obtain Eqs.(4A.1)-(4A.3), and

$$\{xxxx, yyxx, zzxx, xxyy, zzyy\} \cong \text{Eq.(4A.2)},$$

$$\{xxxx, yyxx, zzxx, xxyy, zzzz\} \cong \text{Eq.(4A.3)}.$$

(2) From Eq.(4A.13), we obtain Eqs.(4A.4)-(4A.6), and

$$\{xxxx, yyxx, yxyx, xxyy, xyyx\} \cong \text{Eq.(4A.4)},$$

$$\{xxxx, yyxx, yxyx, xxyy, yyyy\} \cong \text{Eq.(4A.5)}.$$

(3) From Eq.(4A.14), we obtain Eqs.(4A.4), (4A.7), and

$$\{xxxx, yyxx, yxyx, yxxy, xyyx\} \cong \text{Eq.(4A.4)},$$

$$\{xxxx, yyxx, yxyx, yxxy, xyxy\} \cong \text{Eq.(4A.4)},$$

$$\{xxxx, yyxx, yxyx, yxxy, yyyy\} \cong \text{Eq.(4A.4)}.$$

(4) From Eq.(4A.15), we obtain Eq.(4A.8) and

$$\{xxxx, yyxx, yxyx, zzzy, yyyz\} \cong \text{Eq.(4A.6)},$$

$$\{xxxx, yyxx, yxyx, zzzy, yxxz\} \cong \text{Eq.(4A.7)},$$

$$\{xxxx, yyxx, yxyx, zzzy, xyxz\} \cong \text{Eq.(4A.6)},$$

$$\{xxxx, yyxx, yxyx, zzzy, xxyz\} \cong \text{Eq.(4A.6)}.$$

(5) From Eq.(4A.16), we obtain Eqs.(4A.2), (4A.9), and

$$\{xxxx, yyxx, xxyy, yyyy, yxyx\} \cong \text{Eq.(4A.5)},$$

$$\{xxxx, yyxx, xxyy, yyyy, yxxy\} \cong \text{Eq.(4A.5)},$$

$$\{xxxx, yyxx, xxyy, yyyy, xyyx\} \cong \text{Eq.(4A.5)},$$

$$\{xxxx, yyxx, xxyy, yyyy, xyxy\} \cong \text{Eq.(4A.5)},$$

$$\{xxxx, yyxx, xxyy, yyyy, xxzz\} \cong \text{Eq.(4A.2)},$$

$$\{xxxx, yyxx, xxyy, yyyy, yyzz\} \cong \text{Eq.(4A.2)},$$

$$\{xxxx, yyxx, xxyy, yyyy, zzyy\} \cong \text{Eq.(4A.2)}.$$

(6) From Eq.(4A.17), we obtain Eq.(4A.10) and

$$\{xxxx, yyxx, xxyy, zzyy, zzxx\} \cong \text{Eq.(4A.2)},$$

$$\{xxxx, yyxx, xxyy, zzyy, xxzz\} \cong \text{Eq.(4A.3)},$$

$$\{xxxx, yyxx, xxyy, zzyy, yyyy\} \cong \text{Eq.(4A.2)},$$

$$\{xxxx, yyxx, xxyy, zzyy, zzzz\} \cong \text{Eq.(4A.10)}.$$

(7) From Eq.(4A.18), we obtain Eqs.(4A.6), (4A.9), (4A.11), and

$$\{xxxx, yyxx, xxyy, zzzz, yxxy\} \cong \text{Eq.(4A.6)},$$

$$\{xxxx, yyxx, xxyy, zzzz, xyyx\} \cong \text{Eq.(4A.6)},$$

$$\{xxxx, yyxx, xxyy, zzzz, xyxy\} \cong \text{Eq.(4A.6)},$$

$$\{xxxx, yyxx, xxyy, zzzz, zzxx\} \cong \text{Eq.(4A.3)},$$

$$\{xxxx, yyxx, xxyy, zzzz, zzyy\} \cong \text{Eq.(4A.11)},$$

$$\{xxxx, yyxx, xxyy, zzzz, xxzz\} \cong \text{Eq.(4A.3)}.$$

(8) From Eq.(4A.19), we obtain

$$\{xxxx, yyxx, zzyy, zzzz, zzxx\} \cong \text{Eq.(4A.1)},$$

$$\{xxxx, yyxx, zzyy, zzzz, xxyy\} \cong \text{Eq.(4A.11)},$$

$$\{xxxx, yyxx, zzyy, zzzz, xxzz\} \cong \text{Eq.(4A.11)},$$

$$\{xxxx, yyxx, zzyy, zzzz, yyzz\} \cong \text{Eq.(4A.11)},$$

$$\{xxxx, yyxx, xxyy, zzzz, yyyy\} \cong \text{Eq.(4A.11)}.$$

From Table III we find that except for Eqs.(4A.10) and (4A.11), each of Eqs.(4A.1)-(4A.11) has different geometric invariants and hence, they are inequivalent. In order to distinguish Eq.(4A.10) from Eq.(4A.11), we need to use other geometric invariants. Note that, for every subset of three elements in a GHZ-Mermin experiment there corresponds the number of sites at which there is a triad. Those numbers are invariant under (S_1) and (S_2). For example, $\{xxxx, yyzz, zzyy\}$ in Eq.(4A.10) has four triads, while $\{xxxx, zzzz, yyzz\}$ in Eq.(4A.11) has two triads. It is evident that there is no subset of three elements in Eq.(4A.11) possessing four triads. This concludes that Eqs.(4A.10) and (4A.11) are inequivalent. The proof is complete.

It is easy to see that Eqs.(4A.1), (4A.3), (4A.6), (4A.7), (4A.10), and (4A.11) are trivial, whose assigned values are illustrated in Table IV. As follows, we show that Eqs.(4A.2), (4A.5), and (4A.9) are also trivial. Indeed, we note that $(x_1x_2x_3x_4) \times (y_1y_2x_3x_4) \times (x_1x_2y_3y_4) \times (y_1y_2y_3y_4) = 1$. This concludes that for Eq.(4A.2), $\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_5 = 1$ and thus, one can assign $\nu(x_1) = \varepsilon_1, \nu(y_1) = \varepsilon_2, \nu(z_1) = \varepsilon_3, \nu(y_3) = \varepsilon_1\varepsilon_4$, and the remaining ones $\nu(\cdot) = 1$. Similarly, since $(y_1y_2x_3x_4) \times$

TABLE IV: The elements not indicated all take the value $\nu(\cdot) = 1$.

(4A.1)	$\nu(x_1) = \varepsilon_1$ $\nu(y_3) = \varepsilon_1\varepsilon_4$	$\nu(y_1) = \varepsilon_2$ $\nu(z_3) = \varepsilon_1\varepsilon_5$	$\nu(z_1) = \varepsilon_3$
(4A.3)	$\nu(x_1) = \varepsilon_1$ $\nu(y_3) = \varepsilon_1\varepsilon_4$	$\nu(y_1) = \varepsilon_2$ $\nu(z_3) = \varepsilon_2\varepsilon_5$	$\nu(z_1) = \varepsilon_3$
(4A.6)	$\nu(x_1) = \varepsilon_1$ $\nu(y_4) = \varepsilon_1\varepsilon_3\varepsilon_4$	$\nu(y_2) = \varepsilon_2$ $\nu(z_1) = \varepsilon_5$	$\nu(y_3) = \varepsilon_3$
(4A.7)	$\nu(x_1) = \varepsilon_1$ $\nu(y_4) = \varepsilon_4$	$\nu(y_2) = \varepsilon_2$ $\nu(z_1) = \varepsilon_5$	$\nu(y_3) = \varepsilon_3$
(4A.10)	$\nu(x_1) = \varepsilon_1$ $\nu(z_1) = \varepsilon_1\varepsilon_3\varepsilon_4$	$\nu(y_1) = \varepsilon_2$ $\nu(z_3) = \varepsilon_2\varepsilon_5$	$\nu(y_3) = \varepsilon_1\varepsilon_3$
(4A.11)	$\nu(x_1) = \varepsilon_1$ $\nu(z_1) = \varepsilon_2\varepsilon_4\varepsilon_5$	$\nu(y_1) = \varepsilon_2$ $\nu(z_3) = \varepsilon_2\varepsilon_5$	$\nu(y_3) = \varepsilon_1\varepsilon_3$

$(y_1x_2y_3x_4) \times (x_1y_2x_3y_4) \times (x_1x_2y_3y_4) = 1$ in Eq.(4A.5) and $(x_1x_2x_3x_4) \times (y_1y_2x_3x_4) \times (x_1x_2y_3y_4) \times (y_1y_2y_3y_4) = 1$ in Eq.(4A.9) respectively, it is easily concluded that Eqs.(4A.5) and (4A.9) are both trivial.

In the sequel, we prove that both Eqs.(4A.4) and (4A.8) are nontrivial, and the associated states exhibiting 100% violation between quantum mechanics and EPR's local realism are GHZ states.

Theorem: Nontrivial GHZ − Mermin *experiments of five elements for the four-qubit system must equivalently be either* Eq.(4A.4) *or* Eq.(4A.8). *Moreover, the associated states exhibiting an "all versus nothing" contradiction between quantum mechanics and* EPR*'s local realism are* GHZ *states.*

Proof. At first, for $|\varphi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle - |1111\rangle)$ we have

$$x_1x_2x_3x_4|\varphi\rangle = -|\varphi\rangle, \tag{4A.19}$$

$$y_1y_2x_3x_4|\varphi\rangle = |\varphi\rangle, \tag{4A.20}$$

$$y_1x_2y_3x_4|\varphi\rangle = |\varphi\rangle, \tag{4A.21}$$

$$x_1x_2y_3y_4|\varphi\rangle = |\varphi\rangle, \tag{4A.22}$$

$$y_1x_2x_3y_4|\varphi\rangle = |\varphi\rangle. \tag{4A.23}$$

By the GHZ-Mermin argument based on EPR's local realism, one has

$$\nu(x_1)\nu(x_2)\nu(x_3)\nu(x_4) = -1, \tag{4A.24}$$

$$\nu(y_1)\nu(y_2)\nu(x_3)\nu(x_4) = 1, \tag{4A.25}$$

$$\nu(y_1)\nu(x_2)\nu(y_3)\nu(x_4) = 1, \tag{4A.26}$$

$$\nu(x_1)\nu(x_2)\nu(y_3)\nu(y_4) = 1, \tag{4A.27}$$

$$\nu(y_1)\nu(x_2)\nu(x_3)\nu(y_4) = 1. \tag{4A.28}$$

However, Eqs.(4A.24)-(4A.28) are inconsistent, because when we take the product of Eqs.(4A.24) and (4A.26)-(4A.28), the value of the left-hand side is one, while the right-hand side is −1. This concludes that Eq.(4A.4) is nontrivial.

Although the inconsistence of Eqs.(4A.24)-(4A.28) is concluded from Eqs.(4A.24) and (4A.26)-(4A.28), the subset of $\{xxxx, yxyx, xxyy, yxxy\}$ in Eq.(4A.4) is not a GHZ-Mermin experiment of four qubits at whose second site there is only one measurement. On the other hand, there are some states other than GHZ's states satisfying Eqs.(4A.19) and (4A.21)-(4A.23), such as $|\psi\rangle = a(|0000\rangle - |1111\rangle) + b(|0100\rangle - |1011\rangle)$ with $|a|^2 + |b|^2 = 1/2$. However, we will show that the states exhibiting the GHZ-Mermin proof in Eq.(4A.4) are the GHZ states. Thus, Eq.(4A.20) and so $yyxx$ plays a crucial role in the GHZ-Mermin experiment Eq.(4A.4).

Similarly, we can prove that Eq.(4A.8) is also nontrivial and omit the details. In the sequel, we prove that the GHZ state is the unique state with equivalence up to a local unitary transformation which presents the GHZ-Mermin proof in both Eqs.(4A.4) and (4A.8).

To this end, we consider the generic form of Eq.(4A.8) and, suppose $|\varphi\rangle$ is the common eigenstate of five commuting nonlocal spin observables such that

$$A_1A_2A_3A_4|\varphi\rangle = \varepsilon_1|\varphi\rangle, \tag{4A.29}$$

$$A'_1A'_2A_3A_4|\varphi\rangle = \varepsilon_2|\varphi\rangle, \tag{4A.30}$$

$$A'_1A_2A'_3A_4|\varphi\rangle = \varepsilon_3|\varphi\rangle, \tag{4A.31}$$

$$A_1A'_2A'_3A_4|\varphi\rangle = \varepsilon_4|\varphi\rangle, \tag{4A.32}$$

$$A''_1A''_2A''_3A'_4|\varphi\rangle = \varepsilon_5|\varphi\rangle, \tag{4A.33}$$

where $(A, A') = (A, A'') = (A', A'') = 0$. According to GHZ-Mermin's analysis based on EPR's local realism, it is concluded that

$$\nu(A_1)\nu(A_2)\nu(A_3)\nu(A_4) = \varepsilon_1, \tag{4A.34}$$

$$\nu(A'_1)\nu(A'_2)\nu(A_3)\nu(A_4) = \varepsilon_2, \tag{4A.35}$$

$$\nu(A_1')\nu(A_2)\nu(A_3')\nu(A_4) = \varepsilon_3, \tag{4A.36}$$

$$\nu(A_1)\nu(A_2')\nu(A_3')\nu(A_4) = \varepsilon_4, \tag{4A.37}$$

$$\nu(A_1'')\nu(A_2'')\nu(A_3'')\nu(A_4') = \varepsilon_5. \tag{4A.38}$$

When $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = 1$, one can assign $\nu(A_1) = \varepsilon_1, \nu(A_2') = \varepsilon_2, \nu(A_3') = \varepsilon_3, \nu(A_4') = \varepsilon_5$, and the remaining ones $\nu(\cdot) = 1$. Therefore, the necessary condition for $|\varphi\rangle$ presenting a GHZ-Mermin-type proof is

$$\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = -1. \tag{4A.39}$$

On the other hand, suppose Eq.(4A.39) holds, then it is impossible to assign values, either 1 or -1, that satisfy Eqs.(4A.35)-(4A.38) because when take the product of Eqs.(4A.35)-(4A.38), the value of the left hand is equal to ε_5 while the right hand is $-\varepsilon_5$. Thus the condition Eq.(4A.39) is the necessary and sufficient condition for $|\varphi\rangle$ presenting a GHZ-Mermin-type proof. In this case, by changing the signs of local observables A_j and A_j' ($A_1 \to -\varepsilon_1 A_1, A_2' \to \varepsilon_2 A_2', A_3' \to \varepsilon_3 A_3'$, and $A_4' \to \varepsilon_5 A_4'$), we have that

$$A_1A_2A_3A_4|\varphi\rangle = -|\varphi\rangle, \tag{4A.40}$$

$$A_1'A_2'A_3A_4|\varphi\rangle = |\varphi\rangle, \tag{4A.41}$$

$$A_1'A_2A_3'A_4|\varphi\rangle = |\varphi\rangle, \tag{4A.42}$$

$$A_1A_2'A_3'A_4|\varphi\rangle = |\varphi\rangle, \tag{4A.43}$$

$$A_1''A_2''A_3''A_4'|\varphi\rangle = |\varphi\rangle. \tag{4A.44}$$

By Eqs.(2.1)-(2.3), one has that

$$A_jA_j' = -A_j'A_j = iA_j'',$$

$$A_j'A_j'' = -A_j''A_j' = iA_j,$$

$$A_j''A_j = -A_jA_j'' = iA_j',$$

$$A_j^2 = (A_j')^2 = (A_j'')^2 = 1.$$

Hence, A_j, A_j', and A_j'' satisfy the algebraic identities of Pauli's matrices [9]. Therefore, choosing A_j'' representation $\{|0\rangle_j, |1\rangle_j\}$, i.e., $A_j''|0\rangle_j = |0\rangle_j, A_j''|1\rangle_j = -|1\rangle_j$, we have that

$$A_j|0\rangle_j = e^{-i\alpha_j}|1\rangle_j, \quad A_j|1\rangle_j = e^{i\alpha_j}|0\rangle_j,$$

$$A_j'|0\rangle_j = ie^{-i\alpha_j}|1\rangle_j, \quad A_j'|1\rangle_j = -ie^{i\alpha_j}|0\rangle_j,$$

where $0 \le \alpha_j \le 2\pi$. We write $|0100\rangle$, etc., as shorthand for $|0\rangle_1 \otimes |1\rangle_2 \otimes |0\rangle_3 \otimes |0\rangle_4$. Since $\{|\epsilon_1\epsilon_2\epsilon_3\epsilon_4\rangle : \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 = 0, 1\}$ is an orthogonal basis of the four-qubit system. We can uniquely write:

$$|\varphi\rangle = \sum_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4=0,1} \lambda_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}|\epsilon_1\epsilon_2\epsilon_3\epsilon_4\rangle$$

with $\sum |\lambda_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}|^2 = 1$. We define the four-qubit operator

$$\begin{aligned}\mathcal{B} = &-A_1A_2A_3A_4 + A_1'A_2'A_3A_4 \\ &+A_1'A_2A_3'A_4 + A_1A_2'A_3'A_4.\end{aligned}$$

Then by Eqs.(4A.40)-(4A.43), one has that $\mathcal{B}|\varphi\rangle = 4|\varphi\rangle$. This conclude that

$$\mathcal{B}^2|\varphi\rangle = 16|\varphi\rangle. \tag{4A.45}$$

However, a simple computation yields that

$$\mathcal{B}^2 = 4 + 4(A_1''A_2'' + A_1''A_3'' + A_2''A_3'').$$

Then, by using Eq.(4A.45) we conclude that

$$|\varphi\rangle = a|0000\rangle + b|0001\rangle + c|1110\rangle + d|1111\rangle$$

where $a = \lambda_{0000}, b = \lambda_{0001}, c = \lambda_{1110}$, and $d = \lambda_{1111}$. From Eqs.(4A.40) and (4A.44) it is concluded that $a = -de^{i(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}, b = -ide^{i(\alpha_1+\alpha_2+\alpha_3)}$, and $c = ide^{i\alpha_4}$. Therefore

$$\begin{aligned}|\varphi\rangle &= -de^{i(\alpha_1+\alpha_2+\alpha_3)}(e^{i\alpha_4}|0000\rangle + i|0001\rangle) \\ &\quad +d(ie^{i\alpha_4}|1110\rangle + |1111\rangle) \\ &= \frac{1}{\sqrt{2}}(e^{i\theta}|000\rangle|u\rangle + e^{i\phi}|111\rangle|v\rangle)\end{aligned}$$

where $|u\rangle = \frac{1}{\sqrt{2}}(e^{i\alpha_4}|0\rangle + i|1\rangle), |v\rangle = \frac{1}{\sqrt{2}}(ie^{i\alpha_4}|0\rangle + |1\rangle)$, and $0 \le \theta, \phi \le 2\pi$. Since $\langle u|v\rangle = 0$, $|\varphi\rangle$ is a GHZ state.

The proof for Eq.(4A.4) is similar and omitted.

B. The case of six elements

We first characterize all four-qubit GHZ-Mermin experiments of six elements as follows.

Proposition: A $\mathrm{GHZ-Mermin}$ *experiment of six elements for the four-qubit system must equivalently be one of the following forms:*

$$\{xxxx, yyxx, zzxx, xxyy, xxzz, yyyy\}, \tag{4B.1}$$

$$\{xxxx, yyxx, zzxx, xxyy, yyyy, zzyy\}, \tag{4B.2}$$

$$\{xxxx, yyxx, zzxx, xxyy, yyyy, zzzz\}, \tag{4B.3}$$

$$\{xxxx, yyxx, zzxx, xxyy, yyzz, zzzz\}, \tag{4B.4}$$

TABLE V: The numbers in the C line are C-invariants, while the numbers in $1-6$'s lines are R-invariants.

4B	1	2	3	4	5	6	7	8	9
C	4	2	4	4	0	4	0	4	4
1	4	3	3	3	5	4	4	4	2
2	3	3	3	3	4	3	4	3	2
3	2	3	3	3	5	4	4	3	2
4	3	3	2	1	4	3	4	3	2
5	2	3	2	2	4	4	4	3	2
6	2	3	1	2	4	0	4	0	2

$$\{xxxx, yyxx, yxyx, xxyy, yxxy, xyyx\}, \tag{4B.5}$$

$$\{xxxx, yyxx, yxyx, xxyy, yxxy, zzzz\}, \tag{4B.6}$$

$$\{xxxx, yyxx, yxyx, xxyy, xyxy, yyyy\}, \tag{4B.7}$$

$$\{xxxx, yyxx, yxyx, xxyy, xyxy, zzzz\}, \tag{4B.8}$$

$$\{xxxx, yyxx, xxyy, zzyy, yyzz, zzzz\}. \tag{4B.9}$$

Moreover, the geometric invariants of Eqs.(4B.1) $-$ (4B.9) *are illustrated in* Table V.

Proof: By the same argument in the case of five elements, it is concluded that a subset of five elements in a GHZ-Mermin experiment of six elements for the four-qubit system is a four-qubit GHZ-Mermin experiment of five elements. Then, by the Proposition in Section IV.A, a four-qubit GHZ-Mermin experiment of six elements must equivalently be one of the forms

$$\{xxxx, yyxx, zzxx, xxyy, xxzz, \star\star\star\star\}, \tag{4B.10}$$

$$\{xxxx, yyxx, zzxx, xxyy, yyyy, \star\star\star\star\}, \tag{4B.11}$$

$$\{xxxx, yyxx, zzxx, xxyy, yyzz, \star\star\star\star\}, \tag{4B.12}$$

$$\{xxxx, yyxx, yxyx, xxyy, yxxy, \star\star\star\star\}, \tag{4B.13}$$

$$\{xxxx, yyxx, yxyx, xxyy, xyxy, \star\star\star\star\}, \tag{4B.14}$$

$$\{xxxx, yyxx, yxyx, xxyy, zzzz, \star\star\star\star\}, \tag{4B.15}$$

$$\{xxxx, yyxx, yxyx, yxxy, zzzz, \star\star\star\star\}, \tag{4B.16}$$

$$\{xxxx, yyxx, yxyx, zzzy, xyyx, \star\star\star\star\}, \tag{4B.17}$$

$$\{xxxx, yyxx, xxyy, yyyy, zzzz, \star\star\star\star\}, \tag{4B.18}$$

$$\{xxxx, yyxx, xxyy, zzyy, yyzz, \star\star\star\star\}, \tag{4B.19}$$

$$\{xxxx, yyxx, xxyy, zzzz, yyzz, \star\star\star\star\}. \tag{4B.20}$$

(1) From Eq.(4B.10) we obtain Eq.(4B.1), and

$$\{xxxx, yyxx, zzxx, xxyy, xxzz, yyzz\} \cong \text{Eq.(4B.1)},$$

$$\{xxxx, yyxx, zzxx, xxyy, xxzz, zzyy\} \cong \text{Eq.(4B.1)},$$

$$\{xxxx, yyxx, zzxx, xxyy, xxzz, zzzz\} \cong \text{Eq.(4B.1)}.$$

(2) From Eq.(4B.11) we obtain Eqs.(4B.2), (4B.3), and

$$\{xxxx, yyxx, zzxx, xxyy, yyyy, xxzz\} \cong \text{Eq.(4B.1)},$$

$$\{xxxx, yyxx, zzxx, xxyy, yyyy, yyzz\} \cong \text{Eq.(4B.1)}.$$

(3) From Eq.(4B.12) we obtain Eq.(4B.4), and

$$\{xxxx, yyxx, zzxx, xxyy, yyzz, xxzz\} \cong \text{Eq.(4B.1)},$$

$$\{xxxx, yyxx, zzxx, xxyy, yyzz, yyyy\} \cong \text{Eq.(4B.1)},$$

$$\{xxxx, yyxx, zzxx, xxyy, yyzz, zzyy\} \cong \text{Eq.(4B.3)}.$$

(4) From Eq.(4B.13) we obtain Eqs.(4B.5), (4B.6), and

$$\{xxxx, yyxx, yxyx, xxyy, yxxy, xyxy\} \cong \text{Eq.(4B.5)},$$

$$\{xxxx, yyxx, yxyx, xxyy, yxxy, yyyy\} \cong \text{Eq.(4B.5)}.$$

(5) From Eq.(4B.14) we obtain Eqs.(4B.7), (4B.8), and

$$\{xxxx, yyxx, yxyx, xxyy, xyxy, yxxy\} \cong \text{Eq.(4B.5)},$$

$$\{xxxx, yyxx, yxyx, xxyy, xyxy, xyyx\} \cong \text{Eq.(4B.5)}.$$

(6) From Eq.(4B.15) we obtain Eqs.(4B.6), (4B.8), and

$$\{xxxx, yyxx, yxyx, xxyy, zzzz, xyyx\} \cong \text{Eq.(4B.6)},$$

$$\{xxxx, yyxx, yxyx, xxyy, zzzz, yyyy\} \cong \text{Eq.(4B.8)}.$$

(7) From Eq.(4B.16) we obtain Eq.(4B.6) and

$$\{xxxx, yyxx, yxyx, yxxy, zzzz, xyyx\} \cong \text{Eq.(4B.6)},$$

$$\{xxxx, yyxx, yxyx, yxxy, zzzz, xyxy\} \cong \text{Eq.(4B.6)},$$

$$\{xxxx, yyxx, yxyx, yxxy, zzzz, yyyy\} \cong \text{Eq.(4B.6)}.$$

(8) From Eq.(4B.17) we obtain

$$\{xxxx, yyxx, yxyx, zzzy, xyyx, yyyz\} \cong \text{Eq.(4B.6)},$$

$$\{xxxx, yyxx, yxyx, zzzy, xyyx, yxxz\} \cong \text{Eq.(4B.6)},$$

$$\{xxxx, yyxx, yxyx, zzzy, xyyx, xyxz\} \cong \text{Eq.(4B.6)},$$

$$\{xxxx, yyxx, yxyx, zzzy, xyyx, xxyz\} \cong \text{Eq.(4B.6)}.$$

(9) From Eq.(4B.18) we obtain Eq.(4B.3) and

$$\{xxxx, yyxx, xxyy, yyyy, zzzz, yxyx\} \cong \text{Eq.(4B.8)},$$

$$\{xxxx, yyxx, xxyy, yyyy, zzzz, yxxy\} \cong \text{Eq.(4B.8)},$$

$$\{xxxx, yyxx, xxyy, yyyy, zzzz, xyyx\} \cong \text{Eq.(4B.8)},$$

$$\{xxxx, yyxx, xxyy, yyyy, zzzz, xyxy\} \cong \text{Eq.(4B.8)},$$

$$\{xxxx, yyxx, xxyy, yyyy, zzzz, xxzz\} \cong \text{Eq.(4B.3)},$$

$$\{xxxx, yyxx, xxyy, yyyy, zzzz, yyzz\} \cong \text{Eq.(4B.3)},$$

$$\{xxxx, yyxx, xxyy, yyyy, zzzz, zzyy\} \cong \text{Eq.(4B.3)}.$$

(10) From Eq.(4B.19) we obtain Eq.(4B.9) and

$$\{xxxx, yyxx, xxyy, zzyy, yyzz, zzxx\} \cong \text{Eq.(4B.3)},$$

$$\{xxxx, yyxx, xxyy, zzyy, yyzz, xxzz\} \cong \text{Eq.(4B.3)},$$

$$\{xxxx, yyxx, xxyy, zzyy, yyzz, yyyy\} \cong \text{Eq.(4B.1)},$$

$$\{xxxx, yyxx, xxyy, zzyy, yyzz, xxzz\} \cong \text{Eq.(4B.3)}.$$

(11) From Eq.(4B.20) we obtain Eqs.(4B.4), (4B.9) and

$$\{xxxx, yyxx, xxyy, zzzz, yyzz, xxzz\} \cong \text{Eq.(4B.1)},$$

$$\{xxxx, yyxx, xxyy, zzzz, yyzz, yyyy\} \cong \text{Eq.(4B.3)}.$$

From Table V we find that except for Eqs.(4B.3) and (4B.4), each of Eqs.(4A.1)-(4A.9) has different geometric invariants and hence, they are inequivalent. However, $\{xxxx, yyzz, zzyy\}$ in Eq.(4B.3) has four triads, while there is no subset of three elements in Eq.(4B.4) possessing four triads, as noted in the case of Eqs.(4A.10) and (4A.11) this concludes that Eqs.(4B.3) and (4B.4) are inequivalent. The proof is complete.

Theorem: Nontrivial GHZ − Mermin *experiments of six elements for the four-qubit system must equivalently be either* Eq.(4B.5) *or* Eq.(4B.6). *Moreover, the associated states exhibiting an "all versus nothing" contradiction between quantum mechanics and* EPR*'s local realism are* GHZ *states.*

Proof. By the above Proposition, it suffices to show that Eqs.(4B.1)-(4B.4) and (4B.7)-(4B.9) are all trivial, while Eqs.(4B.5) and (4B.6) are both nontrivial.

(1) Since $(xxxx)\times(yyxx)\times(xxyy)\times(yyyy) = 1$, for Eq.(4B.1) we have $\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_6 = 1$. Then, one can assign $\nu(x_1) = \varepsilon_1, \nu(y_1) = \varepsilon_2, \nu(z_1) = \varepsilon_3, \nu(y_3) = \varepsilon_1\varepsilon_4, \nu(z_3) = \varepsilon_1\varepsilon_5$, and the remaining ones $v(\cdot) = 1$.

(2) Since $(xxxx)\times(yyxx)\times(xxyy)\times(yyyy) = 1$ and $(xxxx)\times(zzxx)\times(xxyy)\times(zzyy) = 1$, for Eq.(4B.2) we have $\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_5 = 1$ and $\varepsilon_1\varepsilon_3\varepsilon_4\varepsilon_6 = 1$, respectively. Then, one can assign $\nu(x_1) = \varepsilon_1, \nu(y_1) = \varepsilon_2, \nu(z_1) = \varepsilon_3, \nu(y_3) = \varepsilon_1\varepsilon_4$, and the remaining ones $v(\cdot) = 1$.

(3) Since $(xxxx)\times(yyxx)\times(xxyy)\times(yyyy) = 1$, for Eq.(4B.3) we have $\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_5 = 1$. Then, one can assign $\nu(x_1) = \varepsilon_1, \nu(y_1) = \varepsilon_2, \nu(z_1) = \varepsilon_3, \nu(y_3) = \varepsilon_1\varepsilon_4, \nu(z_3) = \varepsilon_3\varepsilon_6$, and the remaining ones $v(\cdot) = 1$.

(4) Since $(yyxx)\times(zzxx)\times(yyzz)\times(zzzz) = 1$, for Eq.(4B.4) we have $\varepsilon_2\varepsilon_3\varepsilon_5\varepsilon_6 = 1$. Then, one can assign $\nu(x_1) = \varepsilon_1, \nu(y_1) = \varepsilon_2, \nu(z_1) = \varepsilon_3, \nu(y_3) = \varepsilon_1\varepsilon_4, \nu(z_3) = \varepsilon_2\varepsilon_5$, and the remaining ones $v(\cdot) = 1$.

(5) Since $(yyxx)\times(yxyx)\times(xxyy)\times(xyxy) = 1$ and $(xxxx)\times(yxyx)\times(xyxy)\times(yyyy) = 1$, for Eq.(4B.7) we have $\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5 = 1$ and $\varepsilon_1\varepsilon_3\varepsilon_5\varepsilon_6 = 1$, respectively. Then, one can assign $\nu(x_1) = \varepsilon_1, \nu(y_2) = \varepsilon_2, \nu(y_3) = \varepsilon_3, \nu(y_4) = \varepsilon_1\varepsilon_3\varepsilon_4$, and the remaining ones $v(\cdot) = 1$.

(6) Since $(yyxx)\times(yxyx)\times(xxyy)\times(xyxy) = 1$, for Eq.(4B.8) we have $\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5 = 1$. Then, one can assign $\nu(x_1) = \varepsilon_1, \nu(y_2) = \varepsilon_2, \nu(y_3) = \varepsilon_3, \nu(y_4) = \varepsilon_1\varepsilon_3\varepsilon_4, \nu(z_1) = \varepsilon_6$, and the remaining ones $v(\cdot) = 1$.

(7) Since $(xxxx)\times(yyxx)\times(xxyy)\times(zzxx)\times(yyzz)\times(zzzz) = 1$, for Eq.(4B.9) we have $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\varepsilon_6 = 1$. Then, one can assign $\nu(x_1) = \varepsilon_1, \nu(y_1) = \varepsilon_2, \nu(y_3) = \varepsilon_1\varepsilon_3, \nu(z_1) = \varepsilon_1\varepsilon_3\varepsilon_4, \nu(z_3) = \varepsilon_2\varepsilon_5$, and the remaining ones $v(\cdot) = 1$.

Since Eq.(4A.4) is included in Eqs.(4B.5) and (4B.6) and $|\varphi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle - |1111\rangle)$ is a common eigenstate of both Eqs.(4B.5) and (4B.6), it is concluded that Eqs.(4B.5) and (4B.6) are both nontrivial. Moreover, as shown in Section IV.A that the GHZ state is the unique state with equivalence up to a local unitary transformation which presents the GHZ-Mermin proof in Eq.(4A.4), we conclude the same result for Eqs.(4B.5) and (4B.6). This completes the proof.

C. The case of seven elements

We characterize all four-qubit GHZ-Mermin experiments of seven elements as follows.

Proposition: A GHZ − Mermin *experiment of seven elements for the four-qubit system must equivalently be one of the following forms:*

$$\{xxxx, yyxx, zzxx, xxyy, xxzz, yyyy, zzzz\}, \tag{4C.1}$$

TABLE VI: The numbers in the C column are C-invariants, while the numbers in $1-7$'s columns are R-invariants.

	C	1	2	3	4	5	6	7
(4C.1)	4	4	3	3	3	3	2	2
(4C.2)	4	4	3	3	4	2	3	3
(4C.3)	0	6	5	5	5	5	5	5
(4C.4)	4	5	4	5	4	4	4	0
(4C.5)	4	4	4	4	4	4	4	0

$$\{xxxx, yyxx, zzxx, xxyy, xxzz, yyyy, zzyy\}, \quad (4C.2)$$

$$\{xxxx, yyxx, yxyx, xxyy, yxxy, xyyx, xyxy\}, \quad (4C.3)$$

$$\{xxxx, yyxx, yxyx, xxyy, yxxy, xyyx, zzzz\}, \quad (4C.4)$$

$$\{xxxx, yyxx, yxyx, xxyy, xyxy, yyyy, zzzz\}. \quad (4C.5)$$

Moreover, the geometric invariants of Eqs.(4C.1) $-$ (4C.5) *are illustrated in* Table VI.

Proof: As similar as above, a subset of six elements in a GHZ-Mermin experiment of seven elements for the four-qubit system is a four-qubit GHZ-Mermin experiment of six elements. Then, by the Proposition in Section IV.B, a four-qubit GHZ-Mermin experiment of seven elements must equivalently be one of the forms

$$\{xxxx, yyxx, zzxx, xxyy, xxzz, yyyy, \star\star\star\star\}, \quad (4C.6)$$

$$\{xxxx, yyxx, zzxx, xxyy, yyyy, zzyy, \star\star\star\star\}, \quad (4C.7)$$

$$\{xxxx, yyxx, zzxx, xxyy, yyyy, zzzz, \star\star\star\star\}, \quad (4C.8)$$

$$\{xxxx, yyxx, zzxx, xxyy, yyzz, zzzz, \star\star\star\star\}, \quad (4C.9)$$

$$\{xxxx, yyxx, yxyx, xxyy, yxxy, xyyx, \star\star\star\star\}, \quad (4C.10)$$

$$\{xxxx, yyxx, yxyx, xxyy, yxxy, zzzz, \star\star\star\star\}, \quad (4C.11)$$

$$\{xxxx, yyxx, yxyx, xxyy, xyxy, yyyy, \star\star\star\star\}, \quad (4C.12)$$

$$\{xxxx, yyxx, yxyx, xxyy, xyxy, zzzz, \star\star\star\star\}, \quad (4C.13)$$

$$\{xxxx, yyxx, xxyy, zzyy, yyzz, zzzz, \star\star\star\star\}. \quad (4C.14)$$

(1) From Eq.(4C.6) we obtain Eqs.(4C.1), (4C.2), and

$$\{xxxx, yyxx, zzxx, xxyy, xxzz, yyyy, yyzz\} \cong \text{Eq.(4C.2)}.$$

(2) From Eq.(4C.7) we obtain Eq.(4C.2) and

$$\{xxxx, yyxx, zzxx, xxyy, yyyy, zzyy, zzzz\} \cong \text{Eq.(4C.2)},$$

$$\{xxxx, yyxx, zzxx, xxyy, yyyy, zzyy, yyzz\} \cong \text{Eq.(4C.2)}.$$

(3) From Eq.(4C.8) we obtain Eq.(4C.1) and

$$\{xxxx, yyxx, zzxx, xxyy, yyyy, zzzz, yyzz\} \cong \text{Eq.(4C.1)},$$

$$\{xxxx, yyxx, zzxx, xxyy, yyyy, zzzz, zzyy\} \cong \text{Eq.(4C.2)}.$$

(4) From Eq.(4C.9) we obtain

$$\{xxxx, yyxx, zzxx, xxyy, yyzz, zzzz, xxzz\} \cong \text{Eq.(4C.2)},$$

$$\{xxxx, yyxx, zzxx, xxyy, yyzz, zzzz, yyyy\} \cong \text{Eq.(4C.1)},$$

$$\{xxxx, yyxx, zzxx, xxyy, yyzz, zzzz, zzyy\} \cong \text{Eq.(4C.1)}.$$

(5) From Eq.(4C.10) we obtain Eqs.(4C.3), (4C.4), and

$$\{xxxx, yyxx, yxyx, xxyy, yxxy, xyyx, yyyy\} \cong \text{Eq.(4C.3)}.$$

(6) From Eq.(4C.11) we obtain Eq.(4C.4) and

$$\{xxxx, yyxx, yxyx, xxyy, yxxy, zzzz, xyxy\} \cong \text{Eq.(4C.4)},$$

$$\{xxxx, yyxx, yxyx, xxyy, yxxy, zzzz, yyyy\} \cong \text{Eq.(4C.4)}.$$

(7) From Eq.(4C.12) we obtain Eq.(4C.5) and

$$\{xxxx, yyxx, yxyx, xxyy, xyxy, yyyy, yxxy\} \cong \text{Eq.(4C.3)},$$

$$\{xxxx, yyxx, yxyx, xxyy, xyxy, yyyy, xyyx\} \cong \text{Eq.(4C.3)}.$$

(8) From Eq.(4C.13) we obtain Eq.(4C.5), and

$$\{xxxx, yyxx, yxyx, xxyy, xyxy, zzzz, yxxy\} \cong \text{Eq.(4C.4)},$$

$$\{xxxx, yyxx, yxyx, xxyy, xyxy, zzzz, xyyx\} \cong \text{Eq.(4C.4)}.$$

(9) From Eq.(4C.14) we obtain

$$\{xxxx, yyxx, xxyy, zzyy, yyzz, zzzz, zzxx\} \cong \text{Eq.(4C.1)},$$

$$\{xxxx, yyxx, xxyy, zzyy, yyzz, zzzz, xxzz\} \cong \text{Eq.(4C.1)},$$

$$\{xxxx, yyxx, xxyy, zzyy, yyzz, zzzz, yyyy\} \cong \text{Eq.(4C.1)}.$$

From Table VI we find that each of Eqs.(4C.1)-(4C.5) has different geometric invariants and hence, they are all inequivalent. The proof is complete.

Theorem: Nontrivial $\mathrm{GHZ-Mermin}$ *experiments of seven elements for the four-qubit system must equivalently be either* Eq.(4C.3) *or* Eq.(4C.4). *Moreover, the associated states exhibiting an "all versus nothing" contradiction between quantum mechanics and* EPR*'s local realism are* GHZ *states.*

Proof. By the above Proposition, it suffices to show that Eqs.(4C.1), (4C.2), and (4C.5) are all trivial, while Eqs.(4C.3) and (4C.4) are both nontrivial.

(1) Since $(xxxx)\times(yyxx)\times(xxyy)\times(yyyy)=1$ and $(xxxx)\times(zzxx)\times(xxzz)\times(zzzz)=1$, for Eq.(4C.1) we have $\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_6=1$ and $\varepsilon_1\varepsilon_3\varepsilon_5\varepsilon_7=1$, respectively. Then, one can assign $\nu(x_1)=\varepsilon_1,\nu(y_1)=\varepsilon_2,\nu(z_1)=\varepsilon_3,\nu(y_3)=\varepsilon_1\varepsilon_4,\nu(z_3)=\varepsilon_1\varepsilon_5$, and the remaining ones $v(\cdot)=1$.

(2) Since $(xxxx)\times(yyxx)\times(xxyy)\times(yyyy)=1$ and $(xxxx)\times(zzxx)\times(xxyy)\times(zzyy)=1$, for Eq.(4C.2) we have $\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_6=1$ and $\varepsilon_1\varepsilon_3\varepsilon_4\varepsilon_7=1$, respectively. Then, one can assign $\nu(x_1)=\varepsilon_1,\nu(y_1)=\varepsilon_2,\nu(z_1)=\varepsilon_3,\nu(y_3)=\varepsilon_1\varepsilon_4,\nu(z_3)=\varepsilon_1\varepsilon_5$, and the remaining ones $v(\cdot)=1$.

(3) Since $(xxxx)\times(yxyx)\times(xyxy)\times(yyyy)=1$ and $(yyxx)\times(yxyx)\times(xxyy)\times(xyxy)=1$, for Eq.(4C.5) we have $\varepsilon_1\varepsilon_3\varepsilon_5\varepsilon_6=1$ and $\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5=1$, respectively. Then, one can assign $\nu(x_1)=\varepsilon_1,\nu(y_2)=\varepsilon_2,\nu(y_3)=\varepsilon_3,\nu(y_4)=\varepsilon_1\varepsilon_3\varepsilon_4,\nu(z_1)=\varepsilon_7$, and the remaining ones $v(\cdot)=1$.

Since Eq.(4B.5) is included in Eqs.(4C.3) and (4C.4) and $|\varphi\rangle=\frac{1}{\sqrt{2}}(|0000\rangle-|1111\rangle)$ is a common eigenstate of both Eqs.(4C.3) and (4B.4), it is concluded that Eqs.(4C.3) and (4C.4) are both nontrivial. Moreover, as shown in Section IV.B that the GHZ state is the unique state with equivalence up to a local unitary transformation which presents the GHZ-Mermin proof in Eq.(4B.5), we conclude the same result for Eqs.(4C.3) and (4C.4). This completes the proof.

D. The case of eight elements

We characterize all four-qubit GHZ-Mermin experiments of eight elements as follows.

Proposition: A $\mathrm{GHZ-Mermin}$ *experiment of eight elements for the four-qubit system must equivalently be one of the following forms:*

$$\{xxxx, yyxx, zzxx, xxyy, xxzz, yyyy, zzzz, yyzz\}, \tag{4D.1}$$

$$\{xxxx, yyxx, yxyx, xxyy, yxxy, xyyx, xyxy, yyyy\}, \tag{4D.2}$$

$$\{xxxx, yyxx, yxyx, xxyy, yxxy, xyyx, xyxy, zzzz\}. \tag{4D.3}$$

Moreover, the geometric invariants of Eqs.(4D.1) − (4D.3) *are illustrated in* Table VII.

TABLE VII: The numbers in the C column are C-invariants, while the numbers in $1-8$'s columns are R-invariants.

	C	1	2	3	4	5	6	7	8
(4D.1)	4	4	4	3	2	4	3	3	4
(4D.2)	0	6	6	6	6	6	6	6	6
(4D.3)	4	6	5	5	5	5	5	5	0

Proof: As similar as above, a subset of seven elements in a GHZ-Mermin experiment of eight elements for the four-qubit system is a four-qubit GHZ-Mermin experiment of seven elements. Then, by the Proposition in Section IV.C, a four-qubit GHZ-Mermin experiment of eight elements must equivalently be one of the forms

$$\{xxxx, yyxx, zzxx, xxyy, xxzz, yyyy, zzzz, \star\star\star\star\}, \tag{4D.4}$$

$$\{xxxx, yyxx, zzxx, xxyy, xxzz, yyyy, zzyy, \star\star\star\star\}, \tag{4D.5}$$

$$\{xxxx, yyxx, yxyx, xxyy, yxxy, xyyx, xyxy, \star\star\star\star\}, \tag{4D.6}$$

$$\{xxxx, yyxx, yxyx, xxyy, yxxy, xyyx, zzzz, \star\star\star\star\}, \tag{4D.7}$$

$$\{xxxx, yyxx, yxyx, xxyy, xyxy, yyyy, zzzz, \star\star\star\star\}. \tag{4D.8}$$

(1) From Eq.(4D.4) we obtain Eq.(4D.1) and

$$\{xxxx, yyxx, zzxx, xxyy, xxzz, yyyy, zzzz, zzyy\} \cong \text{Eq.(4D.1)}.$$

(2) From Eq.(4D.5) we obtain

$$\{xxxx, yyxx, zzxx, xxyy, xxzz, yyyy, zzyy, zzzz\} \cong \text{Eq.(4D.1)},$$

$$\{xxxx, yyxx, zzxx, xxyy, xxzz, yyyy, zzyy, yyzz\} \cong \text{Eq.(4D.1)}.$$

(3) From Eq.(4D.6) we obtain Eqs.(4D.2) and (4D.3).

(4) From Eq.(4D.7) we obtain Eq.(4D.3) and

$$\{xxxx, yyxx, yxyx, xxyy, yxxy, xyyx, zzzz, yyyy\} \cong \text{Eq.(4D.3)}.$$

(5) From Eq.(4D.8) we obtain

$$\{xxxx, yyxx, yxyx, xxyy, xyxy, yyyy, zzzz, yxxy\} \cong \text{Eq.(4D.3)},$$

$$\{xxxx, yyxx, yxyx, xxyy, xyxy, yyyy, zzzz, xyyx\} \cong \text{Eq.(4D.3)}.$$

From Table VII we find that each of Eqs.(4D.1)-(4D.3) has different geometric invariants and hence, they are all inequivalent. The proof is complete.

Theorem: Nontrivial GHZ − Mermin *experiments of eight elements for the four-qubit system must equivalently be either* Eq.(4D.2) *or* Eq.(4D.3). *Moreover, the associated states exhibiting an "all versus nothing" contradiction between quantum mechanics and* EPR*'s local realism are* GHZ *states.*

Proof. By the above Proposition, it suffices to show that Eq.(4D.1) is trivial, while Eqs.(4D.2) and (4D.3) are both nontrivial.

Indeed, since $(xxxx) \times (yyxx) \times (xxyy) \times (yyyy) = 1, (xxxx) \times (zzxx) \times (xxzz) \times (zzzz) = 1$, and $(xxxx) \times (yyxx) \times (xxzz) \times (yyzz) = 1$, for Eq.(4D.1) we have $\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_6 = 1, \varepsilon_1\varepsilon_3\varepsilon_5\varepsilon_7 = 1$, and $\varepsilon_1\varepsilon_2\varepsilon_5\varepsilon_8 = 1$. Then, one can assign $\nu(x_1) = \varepsilon_1, \nu(y_1) = \varepsilon_2, \nu(z_1) = \varepsilon_3, \nu(y_3) = \varepsilon_1\varepsilon_4, \nu(z_3) = \varepsilon_1\varepsilon_5$, and the remaining ones $v(\cdot) = 1$.

On the other hand, Eq.(4C.3) is included in Eqs.(4D.2) and (4D.3) and $|\varphi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle - |1111\rangle)$ is a common eigenstate of both Eqs.(4D.2) and (4D.3), it is concluded that Eqs.(4D.2) and (4D.3) are both nontrivial. Moreover, as shown in Section IV.C that the GHZ state is the unique state with equivalence up to a local unitary transformation which presents the GHZ-Mermin proof in Eq.(4C.3), we conclude the same result for Eqs.(4D.2) and (4D.3). This completes the proof.

V. MAXIMAL GHZ-MERMIN EXPERIMENTS

In this section, we show that a GHZ-Mermin experiment of the four-qubit system contains at most nine elements and those maximal GHZ-Mermin experiments of nine elements have two different forms, one of which is trivial, while another one is nontrivial.

Proposition: A GHZ − Mermin *experiment of the four-qubit system contains at most nine elements, and a four-qubit* GHZ − Mermin *experiment of nine elements must equivalently be one of the following forms:*

$$\{xxxx, yyxx, zzxx, xxyy, xxzz, yyyy, zzzz, yyzz, zzyy\}, \tag{5.1}$$

$$\{xxxx, yyxx, yxyx, xxyy, yxxy, xyyx, xyxy, yyyy, zzzz\}. \tag{5.2}$$

Moreover, the geometric invariants of Eqs.(5.1) *and* (5.2) *are illustrated in* Table VIII.

TABLE VIII: The numbers in the C column are C-invariants, while the numbers in $1-9$'s columns are R-invariants.

	C	1	2	3	4	5	6	7	8	9
(5.1)	4	4	4	4	4	4	4	4	4	4
(5.2)	4	6	6	6	6	6	6	6	6	0

Proof: Indeed, as similar as above, a subset of eight elements in a GHZ-Mermin experiment of nine elements for the four-qubit system is a four-qubit GHZ-Mermin experiment of eight elements. Then, by the Proposition in Section IV.D, a four-qubit GHZ-Mermin experiment of nine elements must equivalently be one of the forms

$$\{xxxx, yyxx, zzxx, xxyy, xxzz, yyyy, zzzz, yyzz, \star\star\star\star\}, \tag{5.3}$$

$$\{xxxx, yyxx, yxyx, xxyy, yxxy, xyyx, xyxy, yyyy, \star\star\star\star\}, \tag{5.4}$$

$$\{xxxx, yyxx, yxyx, xxyy, yxxy, xyyx, xyxy, zzzz, \star\star\star\star\}. \tag{5.5}$$

From Eq.(5.3) we obtain Eq.(5.1), as well from Eqs.(5.4) and (5.5) obtain Eq.(5.2).

On the other hand, it is evident that one cannot add a element into Eq.(5.1) or (5.2) for obtaining a larger GHZ-Mermin experiment. This completes the proof.

Theorem: Nontrivial GHZ − Mermin *experiments of nine elements for the four-qubit system must equivalently be* Eq.(5.2). *Moreover, the associated states exhibiting an "all versus nothing" contradiction between quantum mechanics and* EPR*'s local realism are* GHZ *states.*

Proof. By the above Proposition, it suffices to show that Eq.(5.1) is trivial, while Eq.(5.2) is nontrivial.

Indeed, since $(xxxx) \times (yyxx) \times (xxyy) \times (yyyy) = 1, (xxxx) \times (zzxx) \times (xxzz) \times (zzzz) = 1, (xxxx) \times (yyxx) \times (xxzz) \times (yyzz) = 1$, and $(xxxx) \times (zzxx) \times (xxyy) \times (zzyy) = 1$, for Eq.(5.1) we have $\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_6 = 1, \varepsilon_1\varepsilon_3\varepsilon_5\varepsilon_7 = 1, \varepsilon_1\varepsilon_2\varepsilon_5\varepsilon_8 = 1$, and $\varepsilon_1\varepsilon_3\varepsilon_4\varepsilon_9 = 1$. Then, one can assign $\nu(x_1) = \varepsilon_1, \nu(y_1) = \varepsilon_2, \nu(z_1) = \varepsilon_3, \nu(y_3) = \varepsilon_1\varepsilon_4, \nu(z_3) = \varepsilon_1\varepsilon_5$, and the remaining ones $v(\cdot) = 1$.

On the other hand, Eq.(4D.2) is included in Eq.(5.2) and $|\varphi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle - |1111\rangle)$ is a common eigenstate of Eq.(5.2), it is concluded that Eq.(5.2) is nontrivial. Moreover, as shown in Section IV.D that the GHZ state is the unique state with equivalence up to a local unitary transformation which presents the GHZ-Mermin proof in Eq.(4D.2), we conclude the same result for Eq.(5.2). This completes the proof.

VI. CONCLUSION

By using some subtle mathematical arguments, we present a complete construction of the GHZ theorem for the four-qubit system. Two geometric invariants play a

crucial role in our argument. We have shown that a GHZ-Mermin experiment of the four-qubit system contains at most nine elements and a four-qubit GHZ-Mermin experiment presenting the GHZ-Mermin-like proof contains at least five elements. We have exhibited all four-qubit GHZ-Mermin experiments of 3-9 elements.

In particular, we have proved that the four-qubit states exhibiting 100% violation between quantum mechanics and EPR's local realism are equivalent to $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0000\rangle - |1111\rangle)$ up to a local unitary transformation, which maximally violate the following Bell inequality

$$\langle\mathcal{B}\rangle \leq 9, \tag{6.1}$$

where $\mathcal{B} = -x_1x_2x_3x_4 + y_1y_2x_3x_4 + y_1x_2y_3x_4 + x_1x_2y_3y_4 + y_1x_2x_3y_4 + x_1y_2y_3x_4 + x_1y_2x_3y_4 - y_1y_2y_3y_4 + z_1z_2z_3z_4$. On the other hand, as shown in Theorem in Sec.V, the state maximally violating Eq.(6.1) is unique and equal to $|\mathrm{GHZ}\rangle$. This yields that the maximal violation of statistical predictions is equivalent to (and so implies) the violation of definite predictions between quantum mechanics and EPR's local realism for the four-qubit system, as similar to the three-qubit system [5]. Therefore, from the view of EPR's local realism one concludes that the maximally entangled states of four qubits should be just the GHZ state [10]. We would like to expect the same result holds true for n qubits, that is, all states exhibiting 100% violation between quantum mechanics and EPR's local realism must be GHZ's states up to a local unitary transformation. This will provides a natural definition of GHZ states and hence clarifies the maximally entangled states of n qubits.

Note that Eq.(6.1) is not a standard Bell inequality which have two observables at each site [11]. From the viewpoint of GHZ's theorem we need to study Bell inequalities of n qubits in which measurements on each particle can be chosen among three spin observables. This should be helpful to reveal the close relationship among entanglement, Bell inequalities, and EPR's local realism [12]. Since the Bell inequalities and GHZ's theorem are two main theme on the violation of EPR's local realism, it turns out that GHZ's theorem and Bell-type inequalities can be used to reveal what the term maximally entangled states should actually mean in the multipartite and/or higher dimensional quantum systems.

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